PRINCIPLES OF ANALYSIS
LECTURE 15 - LIMITS OF FUNCTIONS

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1. Limit of a Function

Let $D \subset \mathbb{R}$ and let $x_0 \in \mathbb{R}$ be an accumulation point of D. Let $f : D \to \mathbb{R}$ and let $L \in \mathbb{R}$. We say that L is the *limit* of f at x_0, and write $L = \lim_{x \to x_0} f(x)$, if

$$\forall \epsilon > 0 \exists \delta > 0 \ni 0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \epsilon.$$

Proposition 1. *Let $D \subset \mathbb{R}$ and let $x_0 \in \mathbb{R}$ be an accumulation point of D. Let $f : D \to \mathbb{R}$ and let $L \in \mathbb{R}$. Then $\lim_{x \to x_0} f(x) = L$ if and only if every deleted neighborhood of x_0 is mapped by f into a neighborhood of L.*

2. Main Examples

Do you know your asymptote from a hole in the graph?

Let $D = (-\infty, 0) \cup (0, \infty)$ and consider these examples of $f : D \to \mathbb{R}$.

Example 1 (Asymptote)**.** Let $f(x) = \frac{1}{x}$.

Example 2 (Hole in the Graph)**.** Let $f(x) = \frac{x^2}{x}$.

Example 3 (Jump)**.** Let $f(x) = \frac{|x|}{x}$.

Example 4 (Oscillation)**.** Let $f(x) = \sin(\frac{1}{x})$.

Example 5 (Squeeze)**.** Let $f(x) = x \sin(\frac{1}{x})$.

Example 6 (Steps)**.** For $k \in \mathbb{Z}^+$, let $D_k = (\frac{1}{2^k}, \frac{1}{2^{k-1}})$. Let $\{y_n\}_{n=1}^{\infty}$ be any sequence of real numbers. Consider the function $f : (0, 1] \to \mathbb{R}$ given by $f(x) = y_k$ if $x \in D_k$.

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3. One-Sided Limits

Let $D \subset \mathbb{R}$ and let $x_0 \in \mathbb{R}$. Let $f : D \to \mathbb{R}$ and let $L \in \mathbb{R}$.

The *left side of D with respect to x_0* is

$$D_{x_0^-} = (-\infty, x_0) \cap D.$$

We say that x_0 is a *left-sided accumulation point* of D if x_0 is an accumulation point of $D_{x_0^-}$.

The *right side of D with respect to x_0* is

$$D_{x_0^+} = (x_0, \infty) \cap D.$$

We say that x_0 is a *right-sided accumulation point* of D if x_0 is an accumulation point of $D_{x_0^+}$.

Clearly, x_0 is an accumulation point of D if and only if x_0 is either a left-sided or right-sided accumulation point of D, or both. We say that x_0 if a *two-sided* accumulation point of D if it is both a left-sided and a right-sided accumulation point of D.

The *left restriction of f with respect to x_0* is $f \restriction_{D_{x_0^-}}$.

The *right restriction of f with respect to x_0* is $f \restriction_{D_{x_0^+}}$.

We say that L is the *left-sided limit* of f at x_0, and write $L = \lim_{x \to x_0^-} f(x)$, if L is the limit of $f \restriction_{D_{x_0^-}}$.

We say that L is the *right-sided limit* of f at x_0, and write $L = \lim_{x \to x_0^+} f(x)$, if L is the limit of $f \restriction_{D_{x_0^+}}$.

If x_0 is a two-sided accumulation point of D, then L is a limit at x_0 if and only if L is both a left-sided and a right-sided limit at x_0.

4. Limits and Sequences

Lemma 1. *Let $\{s_n\}_{n=1}^{\infty}$ be a sequence and let $L \in \mathbb{R}$. Then $\{s_n\}_{n=1}^{\infty}$ converges to L if and only if every subsequence of $\{s_n\}_{n=1}^{\infty}$ converges to L.*

Proof. We already saw this. □

Proposition 2. *Let $f : D \to \mathbb{R}$ and x_0 an accumulation point of D. Then f has a limit at x_0 if and only if for every sequence $\{x_n\}_{n=1}^{\infty}$ in $D \smallsetminus \{x_0\}$ converging to x_0, the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges.*

Proof. We prove both directions.

($\Rightarrow$) Suppose that f has a limit at x_0, and let L be this limit. Let $\epsilon > 0$ and let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $D \smallsetminus \{x_0\}$ which converges to x_0. There exists $\delta > 0$ such that if $0 < |x - x_0| < \delta$, then $|f(x) - L| < \epsilon$. Also, since $\{x_n\}_{n=1}^{\infty}$ converges to x_0 and $x_n \neq x_0$ for all n, there exists $N \in \mathbb{Z}^+$ such that if $n \geq N$, then $0 < |x_n - x_0| < \delta$. Thus, for $n \geq N$, $|f(x_n) - L| < \epsilon$. Thus $\lim_{n\to\infty} f(x_n) = L$, and in particular, $\{f(x_n)\}_{n=1}^{\infty}$ converges.

($\Leftarrow$) Suppose that for every sequence $\{x_n\}_{n=1}^{\infty}$ in $D \smallsetminus \{x_0\}$ converging to x_0, the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges. We wish to show that f has a limit at x_0.

First we claim that if $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are sequence in $D \smallsetminus \{x_0\}$ which converge to x_0, with limits L_1 and L_2 respectively, then $L_1 = L_2$. To see this, form a new sequence $\{z_n\}_{n=1}^{\infty}$ by $z_{2n-1} = x_n$ and $z_{2n} = y_n$. Then $\{z_n\}_{n=1}^{\infty}$ has a limit, say L. Moreover, every subsequence of $\{z_n\}_{n=1}^{\infty}$ converges to L, and $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are subsequences. Thus $L_1 = L_2 = L$.

Now let L denote the common limit of the sequences under consideration; we wish to show that L is the limit of f at x_0. Suppose not. Then there exists $\epsilon > 0$ such that for every $\delta > 0$, there exists x with $0 < |x - x_0| < \delta$ but $|f(x) - L| \geq \epsilon$. For $n \in \mathbb{Z}^+$, let $x_n \in \mathbb{R}$ be and element such that $0 < |x_n - x_0| < \frac{1}{n}$ but $|f(x_n) - L| \geq \epsilon$. Then $\{x_n\}_{n=1}^{\infty}$ converges to x_0, but $\{f(x_n)\}_{n=1}^{\infty}$ does not converge to L. This contradicts the hypothesis. □

Corollary 1. *Let $D \subset \mathbb{R}$ and let x_0 be an accumulation point of D. Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ have limits at x_0. Then so do $f + g$, fg, and f/g when g is nonzero in $D \cap U$ for some neighborhood U of x_0. Moreover,*

- **(a)** $\lim_{x\to x_0}(f(x) + g(x)) = \lim_{x\to x_0} f(x) + \lim_{x\to x_0} g(x)$;
- **(b)** $\lim_{x\to x_0}(f(x) \cdot g(x)) = \lim_{x\to x_0} f(x) \cdot \lim_{x\to x_0} g(x)$;
- **(c)** $\lim_{x\to x_0}(f(x)/g(x)) = \lim_{x\to x_0} f(x)/ \lim_{x\to x_0} g(x)$ *(if appropriate).*

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